JUNIOR CERTIFICATE HIGHER LEVEL

Active Maths 2

Old Syllabus Strand 5

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FOLENS

Contents

The following material has been extracted from *Discovering Maths 2* for Junior Certificate Higher Level.

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Layout: Compuscript

Illustrations: Compuscript, Rory O'Neill

Folens Publishers, Hibernian Industrial Estate, Greenhills Road, Tallaght, Dublin 24, Ireland

Old Syllabus Functions and Graphs

Learning Outcomes

In this chapter you will learn:

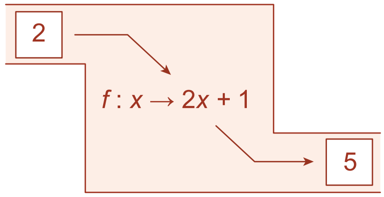
- **About functions, domain, range and codomain**
- **To solve problems based on linear and quadratic graphs**
- **To apply graphs to solve real-life problems**

1.1 FUNCTIONS

YOU SHOULD REMEMBER...

- Functions and graphs from *Active Maths 1*.

A function is like a machine. When a number is put in, it is transformed and a new number emerges. The letter x is often used to denote the input. The letter y is usually used to denote the output. Another name for the output is $f(x)$.

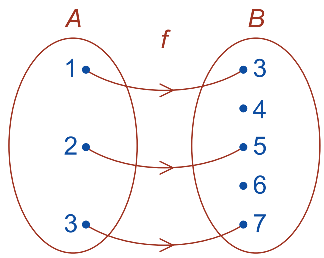


Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5, 6, 7\}$. We can define a function f from A to B in the following way:

$f: A \to B: x \to 2x + 1$

This means that f is a function which maps elements of A onto elements of B, where the output is twice-the-input-plus-one.

Under this function, $1 \to 3$, $2 \to 5$ and $3 \to 7$, as shown in the mapping diagram:



- The set A is called the **domain**. The domain is the set of inputs.
- The set B is called the **codomain**. The codomain is the set of allowable outputs.
- The **range** of f is the set of elements of the codomain which are actually used up. In this case, the range $= \{3, 5, 7\}$. The range, therefore, is a subset of the codomain.

KEY WORDS

- Domain
- Range
- Codomain
- Axes and scales

> f could have been written in the form $y = 2x + 1$ or as the set of couples $\{(1,3), (2,5), (3,7)\}$ or $f(1) = 3$, $f(2) = 5$, $f(3) = 7$.
>
> When a function is written as a set of couples, all the first components will always be different. For example, in the set of couples $\{(1,3), (2,5), (3,7)\}$ the first components are 1, 2, 3 — all different.

Exercise 1.1

1. $f: x \to 4x - 1$. The domain of f is $\{1, 2, 3, 4\}$. Find the range.

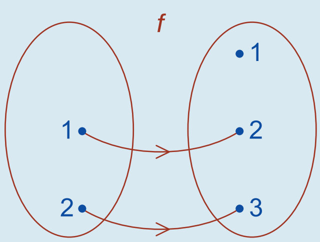
2. $g: x \to 2x^2 + 1$. The domain of g is $\{0, 1, 2\}$. Find the range.

3. List the elements of:

(i) The domain of f

(ii) The codomain of f

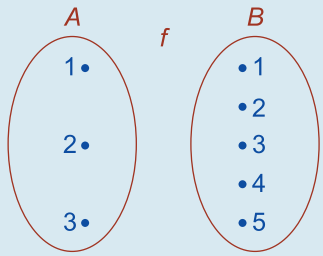
(iii) The range of f



4. $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$ are two sets.

$f: A \to B: x \to 2x - 1$ is a function.

(a) Copy and complete the mapping diagram of f.



(b) Write down:

(i) The domain of f

(ii) The codomain of f

(iii) The range of f

5. The domain of $f: x \to 3x^2 - 1$ is $\{-1, 0, 1\}$. Find the range.

6. $A = \{-1, 0, 1\}$ is a set of numbers.

$f: A \to A: x \to 1 - x^2$ is a function.

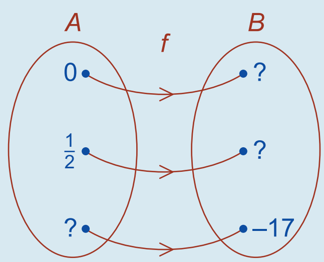
(i) List the couples of f.

(ii) Write down the domain of f.

(iii) Write down the codomain of f.

(iv) Write down the range of f.

7. $f: x \to 3x - 11$ is a function. Copy and complete the mapping diagram of the function f.



8. $g: x \to 4x^2 - 9$ is a function.

(i) Find $g(3)$ and $g(-3)$.

(ii) If $g(k) = 55$, find two possible values of k.

9. $f: x \to 5 - 2x$. The range of f is $\{5, 4, 3, 9\}$. Find the domain of f.

10. $g: x \to (2 - x)(2 + x)$ is defined for $x \in R$.

(i) Find $g(5)$.

(ii) Investigate if $g(2) + g(3) = g(5)$.

(iii) If $g(y) = -45$, find two possible values of y.

(iv) If $g(k) = 3k$, find two possible values of k.

11. $f: x \to \dfrac{1}{x + 2}$

(i) Evaluate $f(3)$ and $f\left(\dfrac{1}{3}\right)$.

(ii) If $f(t) = \dfrac{3}{4}$, find the value of t.

12. $g: x \to \dfrac{x}{x + 1}$ is a function.

(i) Find $g(2)$ and $g\left(\dfrac{1}{2}\right)$.

(ii) If $g(n) = \dfrac{3}{5}$, find the value of n.

(iii) If $g(1) + g(2) = g(k)$, find the value of k.

13. $f: x \to x(x - 2)$

(i) Evaluate $f(3)$.

(ii) Evaluate $f(1)$.

(iii) Find two values of x for which $f(x) = 0$.

(iv) Find two values of y for which $f(y) = 8$.

14. $f(x) = ax + b$. If $f(1) = 5$ and $f(2) = 12$, find a and b.

15. $f: x \to (3 - x)(2x - 3)$

(i) Evaluate $f(2)$.

(ii) Find a value of x (other than 2) for which $f(x) = f(2)$.

16. The domain of $f: x \to x^2 + 3$ is $\{-3, -2, -1, 0, 1, 2, 3\}$. What is the least element in the range?

17. N is the set of natural numbers $= \{1, 2, 3, 4, 5 \ldots\}$.
$f: N \to N: x \to 2x$

Describe in words:

(i) The codomain

(ii) The range

18. $f: x \to 2x^2 - 1$ is defined for all real numbers. What is the least number in the range of f?

19. $f: x \to 5x - 1$.
If $f(2) = kf\left(\dfrac{1}{2}\right)$, find the value of k.

20. $g: x \to 4 - 3x$.

If $g(0) + g(y) = g(1)$, find the value of y.

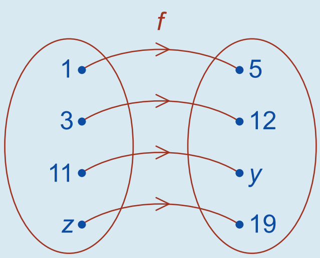
21. $f: x \to x^2 + ax + b$ is a function. If $f(1) = 14$ and $f(2) = 20$, find the values of a and b.

22. $f: x \to 2x^2 + ax + b$ is a function. $f(3) = 12$ and $f(-3) = 18$. Find the value of a and of b.

23. $f: x \to \dfrac{ax + b}{10}$ is a function.

If $f(3) = 3$ and $f(8) = 7$, find the values of a and b. Find, also, the value of $f(5)$.

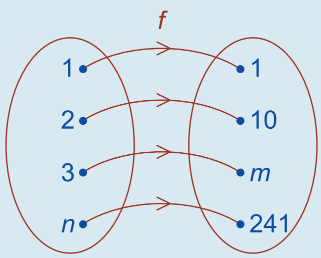
24. The diagram represents a function $f: x \to \dfrac{px + q}{2}$.



(i) Find the values of p and q.

(ii) Find the values of y and z.

25. $f: N \to N: x \to ax^2 + c$ is a function, as illustrated on the diagram.



(i) Find the values of a, c, m and n.

(ii) Write down three natural numbers which are elements of the codomain but not the range.

26. f is a function defined as

$f: R \to R: x \to \dfrac{x^2 - 12}{x}$.

(i) Find $f(6)$.

(ii) Find two values of x for which $f(x) = 1$.

27. f and g are two functions such that $f(x) = x^2 + 2$ and $g(x) = 17 - 2x$.

(i) Find $f(4)$.

(ii) Find $g(4)$.

(iii) Verify that $f(-5) = g(-5)$.

(iv) Find a value of x (other than -5) for which $f(x) = g(x)$.

28. $f(x) = 5x - 2$ is a function.

(i) Evaluate $f(1) - f(-1)$.

(ii) Find the value of k for which $f(k) - f(-k) = 60$.

(iii) Show that $f(x + 1) = 5x + 3$.

29. $f: N \to N: x \to 5x + 3$ is a function.

(i) Investigate if

$f(4) + f(3) = f(7)$.

(ii) Find $f(n + 2)$.

(iii) Find the value of n if

$f(n + 2) = -2$.

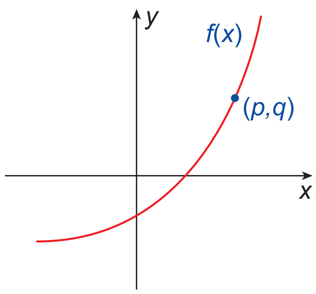
(iv) Write down one number which is an element of the codomain but not the range.

30. $f: x \to x^2 - 8x + b$ is a function.

(i) If $f(5) = 0$, find the value of b.

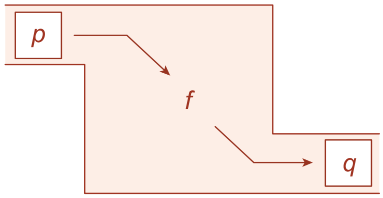
(ii) Find a value of x (other than 5) such that $f(x) = 0$.

1.2 FUNCTIONS AND GRAPHS



If the point (p,q) appears on the graph of some function $y = f(x)$, this means that p is mapped onto q by this function.

i.e. $f(p) = q$



Worked Example 1.1

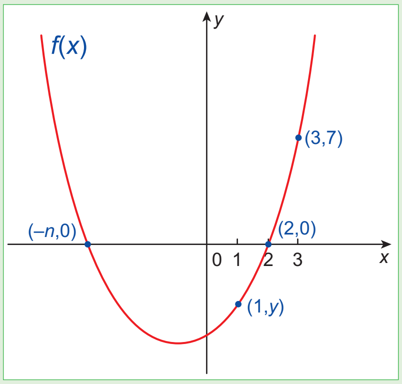
This graph represents the function $f: R \rightarrow R: x \rightarrow x^2 + ax + b$.

(i) Write down two equations in a and b, given that $(2,0)$ and $(3,7)$ are on the graph.

(ii) Find the values of a and b.

(iii) If $(1,y)$ is on the graph, find the value of y.

(iv) If $(-n,0)$ is on the graph, where $n \in N$, find the value of n.



Solution

(i) $(2,0)$ is on the graph $\Rightarrow f(2) = 0$

$\therefore (2)^2 + a(2) + b = 0$

$4 + 2a + b = 0$

$\therefore 2a + b = -4$

$(3,7)$ is on the graph $\Rightarrow f(3) = 7$

$(3)^2 + a(3) + b = 7$

$\therefore 9 + 3a + b = 7$

$\therefore 3a + b = -2$

(ii) Solve the simultaneous equations:

I $\quad 2a + b = -4$

II $\quad 3a + b = -2$

$-1 \times$ I $\quad -2a - b = 4$

II $\quad \underline{3a + b = -2}$

Add $\quad a = 2$

$2a + b = -4$

$\therefore 4 + b = -4$

$\therefore b = -8$

Answer: $a = 2, b = -8$

(iii) We know now that $f(x) = x^2 + 2x - 8$.

If $(1,y)$ is on the graph then $f(1) = y$

$\therefore (1)^2 + 2(1) - 8 = y$

$1 + 2 - 8 = y$

$\therefore -5 = y$ **Answer**

(iv) $(-n,0)$ is on the graph $\Rightarrow f(-n) = 0$

$\therefore (-n)^2 + 2(-n) - 8 = 0$

$n^2 - 2n - 8 = 0$

$(n - 4)(n + 2) = 0$

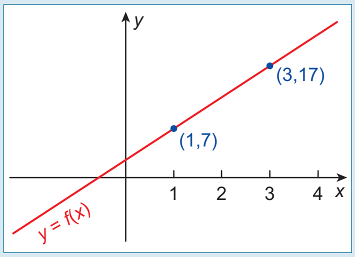
$n = 4$ **OR** $n = -2$

Since $n \in N$, $n = -2$ is rejected.

Answer: $n = 4$

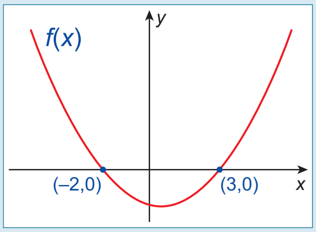
Exercise 1.2

1. $f: x \rightarrow ax + b$ is a function whose graph is illustrated below.



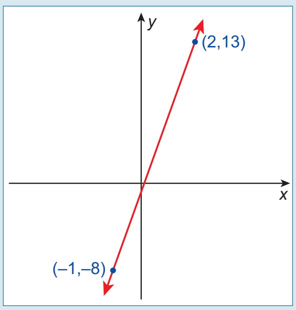
Find the values of a and b.

2. The diagram shows the graph of $f: x \rightarrow x^2 + px + q$, where $p, q \in R$.



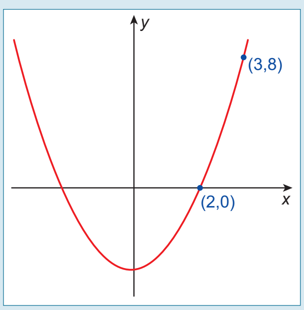
Find the values of a and b.

3. The diagram shows part of a function $y = ax + b$.



Find the values of a and b.

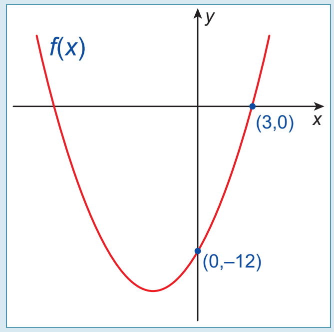
4. The diagram shows part of the graph of the function $y = x^2 + px + q$ where $p, q \in R$.



(i) Find the values of p and q.

(ii) If $(0,n)$ is on the graph, find n.

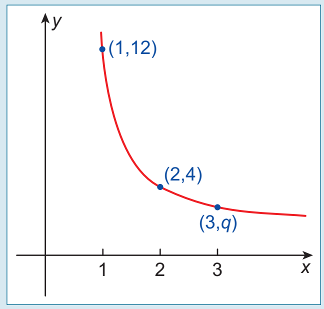
5. The diagram shows part of the graph of the function $f: x \rightarrow x^2 + px + q$ where $\{p, q, x\} \subset R$.



(i) Find the value of p and of q.

(ii) If $(x,0)$ is a point on the graph (where $x \neq 3$), find the value of x.

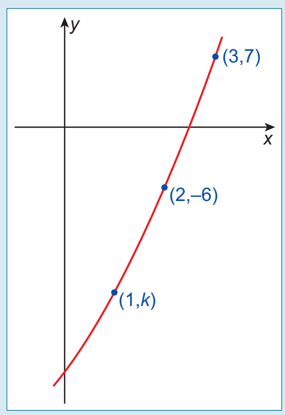
6. The diagram shows part of the graph of $y = \dfrac{12}{ax + b}$.



(i) Find the values of a and b.

(ii) Write q as a fraction.

7. The diagram shows part of the function $y = ax^2 + bx - 20$.

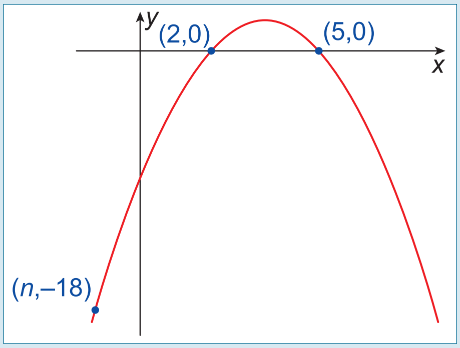


(i) Find the values of a and b.

(ii) Find the value of k.

(iii) Find two values of x for which $y = 0$.

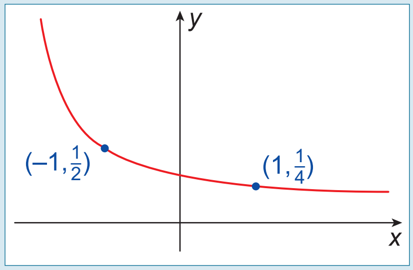
8. The diagram shows part of the function $y = a + bx - x^2$.



(i) Find the values of a and b.

(ii) Find the value of n, where $n < 0$.

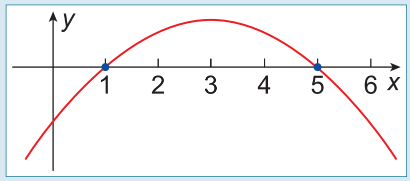
9. The diagram shows part of the graph of the function $f(x) = \dfrac{1}{ax + b}$, where $a, b \in R$.



(i) Find the values of a and b.

(ii) Find the value of t if $f(t) = \dfrac{3}{4}$.

10. The diagram shows part of the graph of $y = a + bx - x^2$.



(i) Find the values of a and b.

(ii) Verify that $f(3 + x) = f(3 - x)$.

(iii) Find two values of x for which $f(3 + x) = 0$.

1.3 GRAPHS

There is a very good case for saying that Sir Isaac Newton (1642–1727) was the greatest mathematician and physicist who ever lived. In physics, he discovered the laws of motion, the laws of gravitation, the laws of light and the laws governing the collisions of spheres. In mathematics, his greatest invention was 'the **calculus**': a method of determining the slope of graphs at any point.

Despite his remarkable discoveries, Newton remained modest about his achievements. He wrote:

> *'I do not know what I may appear to the world, but to myself I seem to have been only a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, while the great ocean of truth lay all undiscovered before me.'*

1.4 LINEAR GRAPHS

The graph of any function of the form $y = ax + b$ (where a and b are constants) is called a **linear graph** because the graph forms a line.

Worked Example 1.2

A car is accelerating uniformly. At time t seconds after it passes a point (p), its velocity (v) in metres per second is given by the formula $v = 12 + 2t$.

(i) Copy and complete the table.

t	0	2	4	6	8	10
v						

(ii) Draw the graph of v for $0 \leqslant t \leqslant 10$.

Estimate from the graph:

(iii) The velocity at $t = 3.6$ seconds

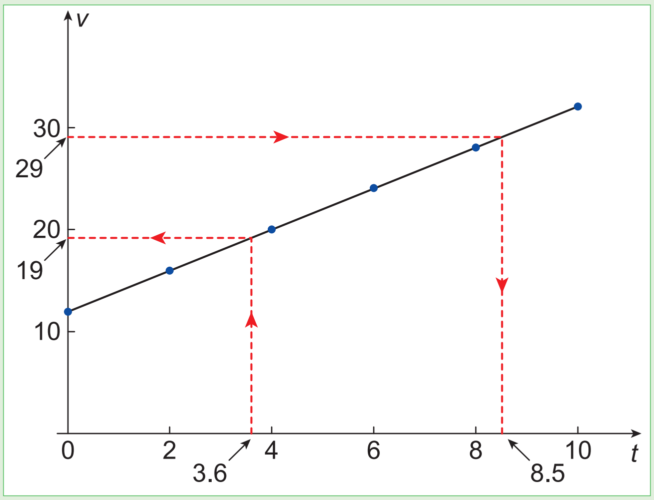
(iv) The time when the velocity is 29 m/s

Solution

(i)

t	0	2	4	6	8	10
v	12	16	20	24	28	32

(ii)



(iii) Draw a line from $t = 3.6$, to the graph and then to the v-axis. The reading is approximately $v = 19$ m/s.

(iv) Draw a line from $v = 29$ to the graph and then to the t-axis. The reading is $t = 8.5$ seconds.

Exercise 1.3

1. Draw the graph of $y = f(x) = 3x - 1$ in the domain $-3 \leqslant x \leqslant 4$, $x \in R$.
Use your graph to estimate:

(i) $f(1.3)$

(ii) The value of x for which $f(x) = -6$

2. Draw the graph of $y = f(x) = 2x - 3$ in the domain $-2 \leqslant x \leqslant 3$, $x \in R$.
Use your graph to estimate:

(i) The value of $f(x)$ when $x = -0.8$

(ii) The value of x for which $f(x) = 0$

3. Using the same scales and axes in the domain $-1 \leqslant x \leqslant 4$, $x \in R$, draw the two graphs $y = 2x - 1$ and $y = 8 - 4x$. Find the point of intersection of the two graphs.

4. A car is travelling at a constant speed of 15 m/s until it passes a point p. The driver then decelerates. The speed (v) in metres per second thereafter is given by $v = 15 - 3t$, where t is the time (in seconds) after the car passes through p.

Draw the graph of v for $0 \leqslant t \leqslant 5$, $t \in R$. Use your graph to estimate:

(i) The speed at $t = 2.3$

(ii) The time when the speed = 10 m/s

(iii) The time when the car stops

5. The time (in minutes) for which a whole salmon should be cooked is given by the formula $t = 20(2m + 1)$, where m = the mass (in kg) of the salmon.

(i) Copy and complete the following table and hence draw the graph.

Mass (kg)	0	1	2	3	4	5	6
Time (mins)	20	60					

(ii) Estimate the time taken to cook a 3.4 kg salmon.

(iii) A salmon is cooked for $1\frac{1}{2}$ hours. What is its mass?

6. The conversion formula for changing from 'gas mark' oven temperature to 'degrees Celsius' is $C = 120 + 14G$, where G = the gas mark and C = degrees Celsius. Draw a conversion graph for $0 \leqslant G \leqslant 6$, $G \in R$.

(i) Estimate the temperature (in degrees Celsius) corresponding to gas mark $2\frac{1}{2}$.

(ii) Estimate the gas mark corresponding to 183° Celsius.

7. $C = \frac{5}{9}(F - 32)$ is the formula which relates the temperature in degrees Fahrenheit (F) to the temperature in degrees Celsius (C).

(i) Copy and complete the following table, giving the values to the nearest integer (whole number).

F	0	20	40	60	80	100
C	−18					

(ii) Draw a linear graph to illustrate this data.

(iii) Estimate the temperature in degrees Celsius when it is 77° Fahrenheit.

(iv) Estimate the temperature in degrees Fahrenheit when it is 15° Celsius.

8. Draw the graphs of the three linear functions:

$f: x \to 2(x - 3)$

$g: x \to 4 - 2x$

$h: x \to \dfrac{3 - 2x}{2}$

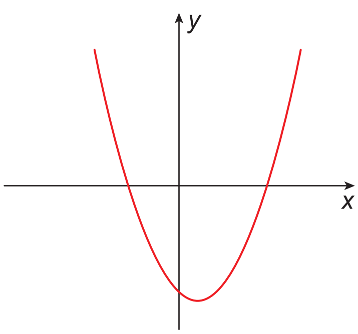
in the domain $-1 \leqslant x \leqslant 4$, $x \in R$.

Show that the three lines are *concurrent* (i.e. that they all pass through the same point).

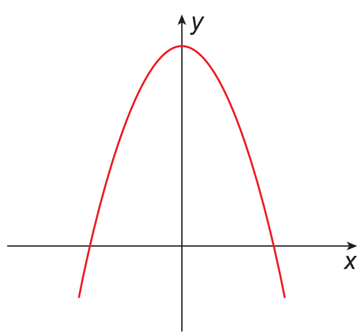
1.5 QUADRATIC GRAPHS

Any graph of the form $y = ax^2 + bx + c$, where $a, b, c \in R$, $a \neq 0$ is called a **quadratic graph**.

If a is a positive number, the graph looks like this:



If a is a negative number, the graph looks like this:



Worked Example 1.3

Draw the graph of the function $f: x \to 3x^2 - 2x - 7$ in the domain $-2 \leqslant x \leqslant 3, x \in R$.
Find, from your graph:

(i) The value of $f(2.5)$

(ii) The values of x for which $f(x) = 3$

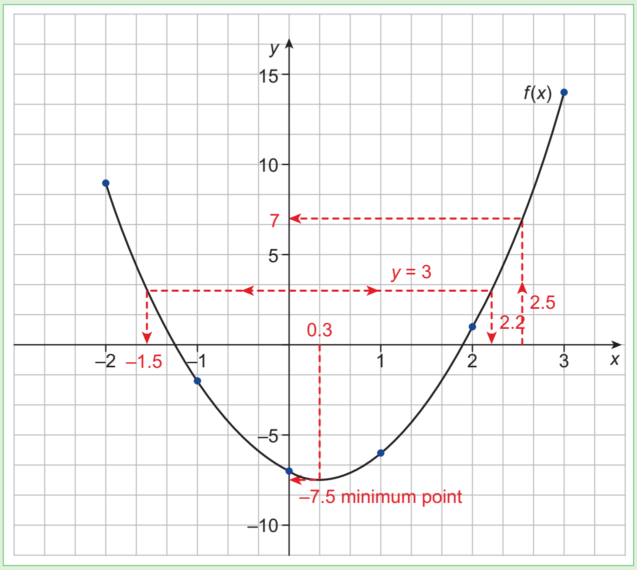
(iii) The minimum value of $f(x)$ and the value of x at which it occurs

(iv) The solution set of $3x^2 - 2x - 7 < 0$

Solution

x	−2	−1	0	1	2	3
$3x^2$	12	3	0	3	12	27
$-2x$	4	2	0	−2	−4	−6
−7	−7	−7	−7	−7	−7	−7
y	9	−2	−7	−6	1	14

Points: $(-2,9)(-1,-2)(0,-7)(1,-6)(2,1)(3,14)$



(i) Draw a line from $x = 2.5$ on the x-axis, up to the graph and across to the y-axis. The reading is approximately 7.
$\therefore f(2.5) = 7$

(ii) Draw lines east and west from $y = 3$ on the y-axis. The corresponding readings on the x-axis are $x = -1.5$ and $x = 2.2$.

(iii) The minimum value of $f(x)$ is approximately -7.5 at $x = 0.3$.

(iv) $3x^2 - 2x - 7 < 0$

$\therefore f(x) < 0$

$\therefore -1.2 < x < 1.9$
(where the graph is below the x-axis)

Worked Example 1.4

Using the same scales and axes, draw the graphs of the functions $f: x \to 4 - 2x - x^2$ and $g: x \to 1 - 2x$ in the domain $-4 \leqslant x \leqslant 3, x \in R$.

Use your graph to estimate:

(i) The range of values of x for which $4 - 2x - x^2 > 0$

(ii) The solutions of the equation $2x^2 + 4x - 5 = 0$

(iii) The value of $\sqrt{3}$

Solution

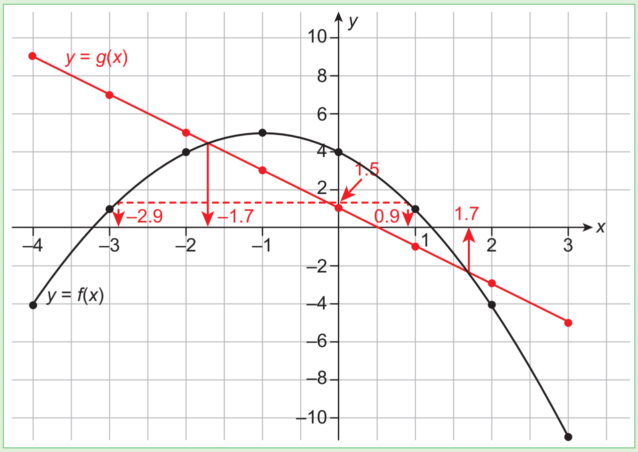
x	-4	-3	-2	-1	0	1	2	3
4	4	4	4	4	4	4	4	4
$-2x$	8	6	4	2	0	-2	-4	-6
$-x^2$	-16	-9	-4	-1	0	-1	-4	-9
$f(x)$	-4	1	4	5	4	1	-4	-11

Points: $(-4, -4)(-3,1)(-2,4)(-1,5)(0,4)(1,1)(2,-4)(3,-11)$

x	-4	-3	-2	-1	0	1	2	3
1	1	1	1	1	1	1	1	1
$-2x$	8	6	4	2	0	-2	-4	-6
$g(x)$	9	7	5	3	1	-1	-3	-5

Points: $(-4,9)(-3,7)(-2,5)(-1,3)(0,1)(1,-1)(2,-3)(3,-5)$

The two graphs are shown.



(i) $4 - 2x - x^2 > 0$

$\therefore f(x) > 0$

$\therefore -3.2 < x < 1.2$ (where the graph of $f(x)$ is above the x-axis)

(ii) Change both sides until you end up with $f(x)$ [i.e. $4 - 2x - x^2$] on one side.

$2x^2 + 4x - 5 = 0$

$x^2 + 2x - 2.5 = 0$ (Dividing both sides by 2)

$2.5 - 2x - x^2 = 0$ (Multiplying both sides by -1)

$4 - 2x - x^2 = 1.5$ (Adding 1.5 to both sides)

$\therefore f(x) = 1.5$

$\therefore x = -2.9$ **or** $x = 0.9$

(iii) $x = \sqrt{3}$

$\therefore x^2 = 3$ (Squaring both sides)

$\therefore 0 = 3 - x^2$ (Subtracting x^2 from both sides)

$\therefore 1 - 2x = 4 - 2x - x^2$ (Adding $(1 - 2x)$ to both sides in order to get $f(x)$ on the right)

$\therefore g(x) = f(x)$

$\therefore$ We want the x-value of the points of intersection of the two graphs. These are $x = -1.7$ and 1.7

Of course, $\sqrt{3}$ is a positive number. $\therefore \sqrt{3} = 1.7$

Exercise 1.4

1. Draw the graph of the function $f: x \to x^2 - 2x - 5$ in the domain $-2 \leqslant x \leqslant 4, x \in R$.
Estimate from your graph:

(i) The value of $f(2.2)$

(ii) The values of x for which $x^2 - 2x - 5 = 0$

(iii) The range of values of x for which $x^2 - 2x - 5 < 0$

(iv) The minimum value of $f(x)$

2. Draw the graph of the function $f: x \to 2x^2 - 3x - 7$ in the domain $-2 \leqslant x \leqslant 3, x \in R$.
Estimate from your graph:

(i) The values of x for which $2x^2 - 3x - 7 = 0$

(ii) The values of x for which $2x^2 - 3x - 2 = 0$

(iii) The range of values of x for which $2x^2 - 3x - 7 \leqslant 0$

(iv) The minimum value of $f(x)$ and the value of x at which it occurs

3. Draw the graph of the function $f: x \to 4 + x - x^2$ in the domain $-2 \leqslant x \leqslant 3, x \in R$.
Use your graph to estimate:

(i) The solution set of $4 + x - x^2 = 0$

(ii) The values of x for which $f(x) > 0$

(iii) The solution set of $1 + x - x^2 = 0$

(iv) The maximum value of $f(x)$

4. Draw the graph of the function $f: x \to 6 - x - 2x^2$ in the domain $-3 \leqslant x \leqslant 3, x \in R$. Estimate from your graph the values of x for which:

(i) $6 = x + 2x^2$

(ii) $2(6 - x^2) = x$

(iii) $6 \geqslant x(2x + 1)$

5. Draw the graph of the function $f: x \to 3x^2 - 3x - 4$ in the domain $-2 \leqslant x \leqslant 3, x \in R$.
Use your graph to estimate:

(i) The values of x for which $3x^2 = 3x + 4$

(ii) The values of x for which $x^2 - x - 5 = 0$

(iii) The range of values of x for which $x^2 - x < 0$

(iv) The minimum value of $f(x)$ and the value of x at which it occurs

6. Draw the graph of the function $f: x \to 4x^2 + 6x - 7$ in the domain $-3 \leqslant x \leqslant 1, x \in R$.
Use your graph to find:

(i) The values of $f(-2.8)$

(ii) The values of x for which $2x^2 + 3x - 3 = 0$

(iii) The range of values of x for which $2x(2x + 3) < 7$

(iv) A negative value of x for which $f(x) = f(1)$

7. Copy and complete the following table for the function $f: x \to 4 - x - 2x^2$.

x	−3	−2	−1	0	1	2	2.5
$f(x)$	−11			4			−11

Draw a graph of $y = f(x)$ in the domain $-3 \leqslant x \leqslant 2.5, x \in R$. Use your graph to find:

(i) The solutions of the equation $2x^2 + x = 4$

(ii) The range of values of x for which $2x^2 + x < 4$

(iii) The maximum value of $f(x)$

(iv) The values of x which satisfy the equation $4x^2 + 2x - 5 = 0$

8. Using the same scales and axes, draw the graphs of the functions $f: x \to x^2 + 2x - 6$ and $g: x \to x - 4$ in the domain $-4 \leqslant x \leqslant 2, x \in R$.
Use your graphs to estimate:

(i) The values of x for which $f(x) = 0$

(ii) The values of x for which $f(x) = g(x)$

(iii) The range of values of x for which $f(x) \leqslant g(x)$

9. Using the same scales and axes, draw the graphs of the functions
$f: x \rightarrow 2 + 2x - x^2$ and $g: x \rightarrow 2x - 3$
in the domain $-2 \leqslant x \leqslant 4, x \in R$.
Estimate from your graphs:

(i) The values of $f(3.5)$ and $g(3.5)$

(ii) The values of x for which
$x^2 = 2(x + 1)$

(iii) The value of $\sqrt{5}$, explaining how you found this answer

10. Using the same scales and axes, draw the graphs of the functions
$f: x \rightarrow 4x^2 + 7x - 3$ and $g: x \rightarrow 2x + 5$
in the domain $-3 \leqslant x \leqslant 1, x \in R$.
Using the graphs, estimate:

(i) The value of x for which $g(x) = 0$

(ii) The values of x for which $f(x) = 0$

(iii) The values of x for which $f(x) = g(x)$

(iv) The range of values of x for which $g(x) > f(x)$

11. Draw the graph of the function
$f: x \rightarrow 5 - 3x - x^2$
in the domain $-5 \leqslant x \leqslant 2, x \in R$.
Using your graph:

(i) Estimate the maximum value of $f(x)$.

(ii) Draw the axis of symmetry of the graph and write down its equation in the form $x = k$.

(iii) Find the values of $f(k + 2)$ and $f(k - 2)$.

(iv) By drawing the graph of the line $y = x$, find the values of x for which $f(x) = x$.

12. The function $f: x \rightarrow 8 + x - x^2$ is defined in the domain $-3 \leqslant x \leqslant 4, x \in R$.

(i) Copy and complete the table.

x	−3	−2	−1	0	1	2	3	4
***f*(*x*)**	−4					6		−4

(ii) Draw the graph $y = f(x)$.

(iii) Estimate the values of x for which $4 = (3 + x)(4 - x)$, using your graph.

(iv) Write down the equation of the axis of symmetry of the graph.

(v) Find a positive value of x for which $f(x) = f(-0.5)$.

(vi) Use the same scales and axes to draw the graph of $g: x \rightarrow x + 1$ and hence estimate $\sqrt{7}$.

1.6 REAL-LIFE PROBLEMS SOLVED BY GRAPHS

Some of life's problems can be solved by drawing a graph. Here is an example of such a problem.

Worked Example 1.5

A family has a small garden. There is a straight wall along one side. Their daughter, Theresa, wants to make a flower-bed. Her parents say that she can make a rectangular flower-bed – using the wall as one side – if the length of the other three sides is exactly 6 metres in total.

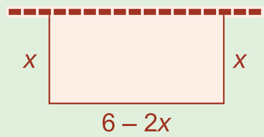
Show that if x is the width of the flower-bed, then the area (A) is given by the expression $A = 6x - 2x^2$.

Draw the graph of $A = 6x - 2x^2$ for $0 \leqslant x \leqslant 3, x \in R$, and hence find the maximum possible area for the flower-bed.

Solution

If x = the width, then

$(6 - 2x)$ = the length.



A = length × width

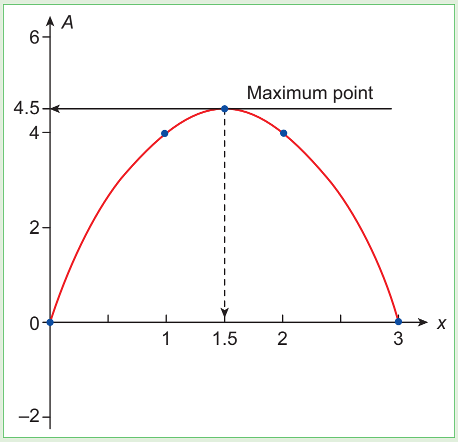
$= (6 - 2x)x$

$= 6x - 2x^2$ **QED**

x	0	1	2	3
$6x$	0	6	12	18
$-2x^2$	0	−2	−8	−18
A	0	4	4	0

Points: (0,0)(1,4)(2,4)(3,0)

The maximum area is 4.5 m² when $x = 1.5$ m.



Exercise 1.5

1. Graph the function $f: x \rightarrow 15x - 3x^2$ in the domain $0 \leqslant x \leqslant 5, x \in R$.

$f(x)$ represents the height (in metres) of a football kicked from level ground, while x represents the time (in seconds) after it is kicked until it hits the ground again.

Use the graph to estimate:

(i) The maximum height reached

(ii) The height of the football after 1.3 seconds

(iii) The time which the football spends in the air before it lands

(iv) The number of seconds the football is more than 9 metres off the ground

2. The depth of water (d) in a harbour is given by the formula $d(x) = 2x^2 + 5x + 7$ where x represents the time (in hours).

$x = 0$ is noon, $x = 1$ is 1 p.m., $x = 2$ is 2 p.m. etc.

Draw the graph of $d(x)$ in the domain $-4 \leqslant x \leqslant 2, x \in R$.

Use your graph to estimate:

(i) The minimum depth in the harbour and the time it occurs

(ii) The times when the water is 10 metres in depth

(iii) The depth at 1.45 p.m.

(iv) The length of time when the depth is less than 12 metres

3. Using the same axes and the same scales, graph the two functions ($x \in R$):

$f: x \rightarrow 15 - x - 2x^2$ in the domain $-3 \leqslant x \leqslant 2.5$

$g: x \rightarrow 6x - x^2$ in the domain $0 \leqslant x \leqslant 3$

$f(x)$ represents the height (in kilometres) of a foreign rocket which is launched at 4.30 p.m. ($x = -3$).

$g(x)$ is the height (in kilometres) of a missile which is launched at 5.00 p.m. ($x = 0$) to intercept the foreign rocket.

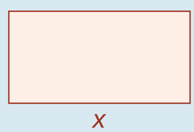
Use the graphs to estimate:

(i) The maximum height reached by the foreign rocket

(ii) The time at which the missile intercepts the rocket

(iii) The height at which the collision occurs

4. The perimeter of a rectangular room is 12 metres. If x = the length, show that the area (A) is given by $A(x) = 6x - x^2$.



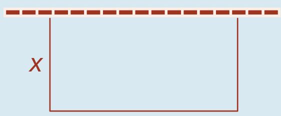
By drawing a graph of $x \to 6x - x^2$ in the domain $0 \leqslant x \leqslant 6, x \in R$, estimate:

(i) The maximum area possible

(ii) The length of the room if the area is 8.5 square metres

5. A straight wall runs through a farm. A farmer has 20 metres of fencing to make a rectangular pig-pen, using the wall as one of the sides.

If x = the width of the pen (in metres), show that the area (A) of the pen is given by the function $A(x) = 20x - 2x^2$.



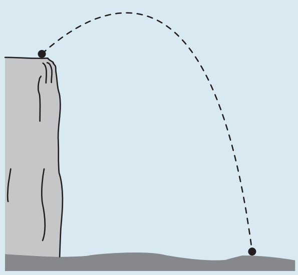
(i) Copy and complete this table and draw the graph of $y = A(x)$ in the domain $0 \leqslant x \leqslant 10, x \in R$.

x	0	1	2	3	4	5	6	7	8	9	10
$A(x)$	0	18	32						32	18	0

(ii) Use your graph to estimate the width if the area is 25 m^2.

(iii) What is the maximum area for the pen and what are its dimensions?

6. A boy throws a stone from the top of a cliff out to sea. The height h (in metres) of the stone above sea level is given by
$h(t) = 18 + 27t - 5t^2$,
where t is the time (in seconds) after it is thrown.



Draw the graph of h in the domain $0 \leqslant t \leqslant 6, t \in R$.

From the graph, calculate:

(i) The height of the cliff

(ii) The maximum height and the time when it is reached

(iii) The time taken for the stone to reach the sea

(iv) The range of t for which $25 < h < 40$

7. The sum of two numbers is 10.
Let x = the first number.

(i) Write down an expression for the other number.

(ii) Find an expression for the sum (S) of the squares of the two numbers in the form $S(x) = ax^2 + bx + c$.

(iii) Complete the table and hence draw the graph of $y = S(x)$ in the domain $0 \leqslant x \leqslant 10, x \in R$.

x	0	2	4	6	8	10
$S(x)$	100	68				

(iv) From your graph find the minimum value for the sum of the squares of the two numbers.

(v) Estimate the values of x for which $S(x) = 80$.

8. A woman buys apples for 10 cents each. She then sells them in the street. If she sells them for 22 cents each, she can sell eight every hour. For each cent by which she reduces the price, she sells two more apples per hour.

(i) If the selling price is 20 cents, calculate the profit which the woman makes per hour.

(ii) If she reduces the selling price by x cents, show that the profit (p) per hour is given by the expression:

$p(x) = 96 + 16x - 2x^2$

(iii) Complete the table and hence draw a graph of $y = p(x)$ in the domain $0 \leqslant x \leqslant 12, x \in R$.

x	0	2	4	6	8	10	12
$p(x)$		120			96		0

(iv) Find the maximum profit per hour and the selling price which yields this profit.

Revision Exercises

1. (a) $f: x \to x^2 - 4x$ is a function. The domain of f is $\{1, 2, 3, 4\}$. Find the range of f.

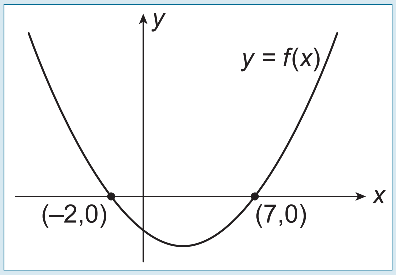
(b) $f: x \to 6 - x^2$ is a function.

(i) Evaluate $f(3)$ and $f(6)$.

(ii) Find the value of k if $f(6) = kf(3)$.

(iii) Find two values of x for which $f(x) = x$.

(c) The diagram shows part of the graph of the function $y = x^2 - ax - b$ where $a, b \in N$.



(i) Find the values of a and b.

(ii) If $(1, n)$ is also a point on the graph, find the value of n.

2. (a) $f: N \to R: x \to \sqrt{5x + 1}$ is a function.

(i) Find the value of $f(7)$.

(ii) If $f(n) = 9$, find the value of n.

(b) $f: x \to x^2 - 3$ is a function defined on R (real numbers).

(i) Evaluate $f(1)$ and $f(2)$.

(ii) Investigate if $f(1) + f(2) = f(3)$.

(iii) If $f(4) + f(-6) = f(n)$, find the two possible values of n.

(c) $f: x \to \dfrac{1}{x - 2}$ is defined for all $x \in R \setminus \{2\}$.

(i) Calculate $f(4)$ and $f\left(\dfrac{1}{4}\right)$ as fractions.

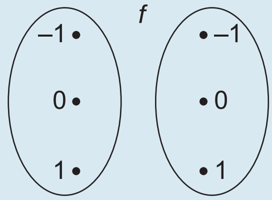
(ii) If $f(k) = -\dfrac{2}{3}$, find the value of $k \in R$.

(iii) Show that

$$f(x) - f\left(\frac{1}{x}\right) = \frac{(x - 1)(x + 1)}{(x - 2)(2x - 1)}.$$

3. (a) $A = \{-1, 0, 1\}$

$f: A \to A: x \to 1 - 2x^2$ is a function.



Copy and complete the graph of f.

List the elements of:

(i) The domain of f

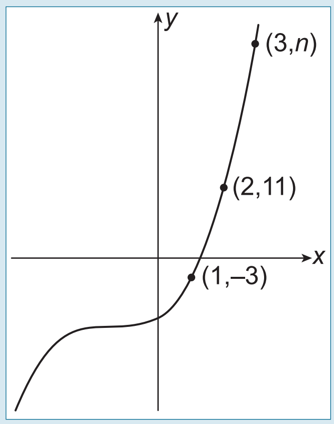
(ii) The codomain of f

(iii) The range of f

(b) $f: R \to R: x \to \sqrt{ax + b}$ is a function.

If $f(1) = 4$ and $f(4) = 7$, find the values of a and b.

(c) The diagram shows part of the graph of $y = px^3 + q$.



(i) Find the value of p and q.

(iii) Evaluate n.

4. Using the same scales and the same axes, draw the graphs of the two functions: $f: x \to 5 - 5x - x^2$ and $g: x \to -x$ in the domain $-6 \leqslant x \leqslant 1$, $x \in R$.

Use your graph to estimate:

(i) The maximum value of $f(x)$ and the values of x at which it occurs

(ii) The values of x for which $f(x) = 0$

(iii) The values of x for which $f(x) = g(x)$

5. Using the same scales and the same axes, draw the graphs of the two functions $f: x \rightarrow 8 + 2x - x^2$ and $g: x \rightarrow 2x + 1$ in the domain $-3 \leqslant x \leqslant 5, x \in R$.

Use your graph to estimate:

(i) The range of values for x for which $8 + 2x - x^2 > 0$

(ii) The solutions of the equation $5 + 2x - x^2 = 0$

(iii) The value of $\sqrt{7}$

6. Draw the graph of the function $f: x \rightarrow 2x^2 - 10x + 7$, in the domain $0 \leqslant x \leqslant 5, x \in R$.

(i) Show the axis of symmetry of the graph and write down its equation in the form $x = k$.

(ii) Find a value of x (other than 1.3) for which $f(x) = f(1.3)$.

(iii) Estimate the range of values of x for which $2x^2 > 10x - 7$.

(iv) Estimate the range of values of x for which $-1 < f(x) < 2$.

7. (a) Find the solutions of $3 - x - x^2 = 0$ correct to one decimal place, using the quadratic formula $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

(b) Draw the graphs of $f: x \rightarrow 3 - x - x^2$ and $g: x \rightarrow 2x - 1$ on the same scales and axes in the domain $-4 \leqslant x \leqslant 3, x \in R$.

(c) Use your graphs to find the values of x for which:

(i) $f(x) > 0$

(ii) $g(x) = f(x)$

(iii) $g(x) \leqslant f(x)$

8. On the same scales and axes draw the graphs of $f: x \rightarrow 28 - x - 2x^2$ in the domain $-4 \leqslant x \leqslant 3.5, x \in R$, and $g: x \rightarrow 17x - 4x^2$ in the domain $0 \leqslant x \leqslant 4.25$, $x \in R$.

$f(x)$ represents the height (in km) of a missile which is launched at 8.00 a.m. ($x = -4$).

$g(x)$ represents the height of a shell which is launched to intercept the missile at 8.20 a.m. ($x = 0$).

x represents the time; each unit represents five minutes.

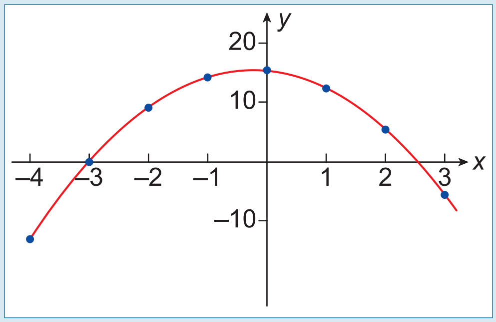
Use the graphs to find:

(i) The maximum height of the missile

(ii) The length of time for which the missile is more than 20 km above the ground

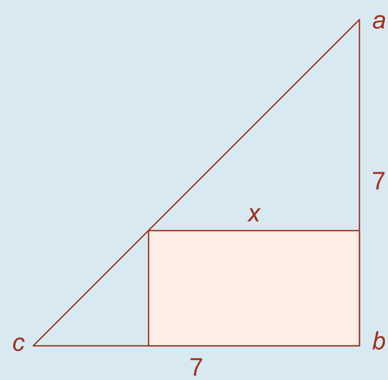
(iii) The time of the collision and the height at which it occurs

9. (a) The diagram shows part of the graph of $y = 15 - x - 2x^2$.



Use the diagram to estimate the values of x for which $15 - x - 2x^2 > 0$.

(b) abc is a triangle which is right angled at b, such that $|ab| = |bc| = 7$.



A rectangle is to be inset, as shown, in the triangle. If x is the length of the rectangle, show that the area (A) is given by $A(x) = 7x - x^2$.

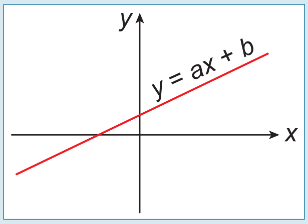
By drawing the graph of $y = A(x)$ in the domain $0 \leqslant x \leqslant 7, x \in R$, find:

(i) The values of x for which the area is 8 square units

(ii) The maximum area and the value of x which gives rise to it

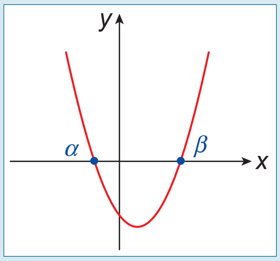
Summary of important points

1. A **linear graph** is the graph of any function of the form $y = ax + b$. The graph will be a straight line.



2. A **quadratic graph** is the graph of any function of the form $y = ax^2 + bx + c$, $a \neq 0$.

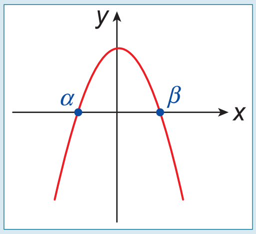
(i) If a is **positive**, the graph looks like this:



$ax^2 + bx + c > 0$ for $x < \alpha$ or $x > \beta$

$ax^2 + bx + c < 0$ for $\alpha < x < \beta$

(ii) If a is **negative**, the graph looks like this:



$ax^2 + bx + c > 0$ for $\alpha < x < \beta$

$ax^2 + bx + c < 0$ for $x < \alpha$ or $x > \beta$

Answers

Exercise 1.1

1. {3, 7, 11, 15} **2.** {1, 3, 9} **3.** (i) {1, 2} (ii) {1, 2, 3} (iii) {2, 3} **4.** (b) (i) {1, 2, 3} (ii) {1, 2, 3, 4, 5} (iii) {1, 3, 5} **5.** {–1, 2} **6.** (i) {(–1,0), (0,1), (1,0)} (ii) {–1, 0, 1} (iii) {–1, 0, 1} (iv) {0, 1} **8.** (i) 27, 27 (ii) 4, −4 **9.** $\{0, \frac{1}{2}, 1, -2\}$ **10.** (i) –21 (ii) No; $0 + (-5) \neq -21$ (iii) ±7 (iv) 1, −4 **11.** (i) $\frac{1}{5}, \frac{3}{7}$ (ii) $-\frac{2}{3}$ **12.** (i) $\frac{1}{3}, \frac{2}{3}$ (ii) $n = \frac{3}{2}$ (iii) $k = -7$ **13.** (i) 3 (ii) –1 (iii) 0, 2 (iv) 4, −2 **14.** $a = 7$, $b = -2$ **15.** (i) 1 (ii) $2\frac{1}{2}$ **16.** 3 (when $x = 0$) **17.** (i) Natural numbers (ii) Even natural numbers **18.** –1 (when $x = 0$) **19.** $k = 6$ **20.** $\frac{7}{3}$ **21.** $a = 3$, $b = 10$ **22.** $a = -1, b = -3$ **23.** $a = 8, b = 6$; 4.6 **24.** (i) $p = 7, q = 3$ (ii) $y = 40, z = 5$ **25.** (i) $a = 3$, $c = -2, m = 25, n = 9$ (ii) Any number except 1, 10, 25, etc. (the range) **26.** (i) 4 (ii) 4, −3 **27.** (i) 18 (ii) 9 (iv) 3 **28.** (i) 10 (ii) 6 **29.** (i) No (ii) $5n + 13$ (iii) −3 (iv) Any number except 3, 8, 13, 18, ..., 30 **30.** (i) 15 (ii) 3

Exercise 1.2

1. $a = 5, b = 2$ **2.** $p = -1, q = -6$ **3.** $a = 7, b = -1$ **4.** (i) $p = 3, q = -10$ (ii) −10 **5.** (i) $p = 1, q = -12$ (ii) $x = -4$ **6.** (i) $a = 2, b = -1$ (ii) $\frac{12}{5}$ **7.** (i) $a = 2$, $b = 3$ (ii) $k = -15$ (iii) 2.5, −4 **8.** (i) −10, 7 (ii) −1 **9.** (i) $a = 1, b = 3$ (ii) $-\frac{5}{3}$ **10.** (i) $a = -5, b = 6$ (iii) 2, −2

Exercise 1.3

1. (i) 3 (ii) −1.7 **2.** (i) −4.6 (ii) 1.5 **3.** $\left(1\frac{1}{2}, 2\right)$ **4.** (i) 8 m/s (ii) 1.7 s (iii) 5 s **5.** (ii) 156 mins (iii) 1.75 kg **6.** (i) 155° (ii) $4\frac{1}{2}$ **7.** (iii) 25°C (iv) 59°F

Exercise 1.4

1. (i) −4.6 (ii) −1.4, 3.4 (iii) $-1.4 < x < 3.4$ (iv) −6 **2.** (i) −1.3, 2.8 (ii) −0.5, 2 (iii) $-1.3 \leqslant x \leqslant 2.8$ (iv) −8.1 at $x = 0.75$ **3.** (i) −1.6, 2.6 (ii) $-1.6 < x < 2.6$ (iii) −0.6, 1.6 (iv) 4.25 **4.** (i) −2, 1.5 (ii) −2.7, 2.2 (iii) $-2 \leqslant x \leqslant 1.5$ **5.** (i) −0.7, 1.7 (ii) $x = -1.8, 2.8$ (iii) $0 < x < 1$ (iv) −4.8 at $x = 0.5$ **6.** (i) 7.5 (ii) $x = -2.2, 0.6$ (iii) $-2.3 < x < 0.7$ (iv) −2.6 **7.** (i) −1.7, 1.2 (ii) $-1.7 < x < 1.2$ (iii) 4.1 (iv) $x = -1.4, 0.9$ **8.** (i) −3.7, 1.7 (ii) −2, 1 (iii) $-2 \leqslant x \leqslant 1$ **9.** (i) −3.3, 4 (ii) −0.7, 2.7 (iii) 2.2 **10.** (i) −2.5 (ii) −2.1, 0.3 (iii) −2.2, 0.9 (iv) $-2.2 < x < 0.9$ **11.** (i) 7.5 (ii) $x = -1.5$ (iii) 3.25 and 3.25 (iv) −5, 1 **12.** (iii) $x = -2.3, 3.3$ (iv) $x = \frac{1}{2}$ (v) 1.5 (vi) $x = 2.65$

Exercise 1.5

1. (i) 18.75 (ii) 14.4 (iii) 5 seconds (iv) 3.6 seconds **2.** (i) 3.75 metres at 10.45 a.m. (ii) 9 a.m. and 12.25 p.m. (iii) 20 metres (iv) 4 hours **3.** (i) 15 km (ii) 5.17 p.m. (iii) 7.5 km **4.** (i) 9 m^2 (ii) 2.3 or 3.7 **5.** (i) 1.5 or 8.5 (ii) 50 m^2; 5 m × 10 m **6.** (i) 18 m (ii) 54.5 at $t = 2.7$ (iii) 6 seconds (iv) $0.3 < t < 1$; $4.5 < t < 5.2$ **7.** (i) $10 - x$ (ii) $2x^2 - 20x + 100$ (iv) 50 (v) 1.1 and 8.9 **8.** (i) 120 cent (ii) $96 + 16x - 2x^2$ (iv) €1.28 at a price of 18 cent ($x = 4$ cent reduction)

Revision Exercises

1. (a) {−3, −4, 0} (b) (i) −3, −30 (ii) 10 (iii) −3, 2 (c) (i) $a = 5, b = 14$ (ii) $n = -18$ **2.** (a) (i) 6 (ii) 16 (b) (i) −2, 1 (ii) No; $(-2) + 1 \neq 6$ (iii) $n = \pm 7$ (c) (i) $\frac{1}{2}, -\frac{4}{7}$ (ii) $\frac{1}{2}$ **3.** (a) (i) {−1, 0, 1} (ii) {−1, 0, 1} (iii) {−1, 1} (b) $a = 11, b = 5$ (c) (i) $p = 2, q = -5$ (ii) $n = 49$ **4.** (i) 11.25 at $x = -25$ (ii) −5.85, 0.85 (iii) −5, 1 **5.** (i) $-2 < x < 4$ (ii) −1.4, 3.4 (iii) 2.6 **6.** (i) $x = 2.5$ (ii) 3.7 (iii) $x < 0.8, x > 4.2$ (iv) $0.5 < x < 1$ or $4 < x < 4.5$ **7.** (a) −2.3, 1.3 (c) (i) $-2.3 < x < 1.3$ (ii) −4, 1 (iii) $-4 \leqslant x \leqslant 1$ **8.** (i) 29 km (ii) 20 mins (iii) 8.30 a.m. at 18 km **9.** (a) $-3 < x < 2.5$ (b) (i) 1.4, 5.6 (ii) 12.25 at $x = 3.5$

This booklet covers Strand 5 of the old syllabus at Junior Certificate Higher Level.

Strand 5 of the new Project Maths syllabus at Higher Level is covered in *Active Maths 2*.

- Students sitting the Junior Certificate exam in **2014** should use this booklet for Strand 5 of the Higher Level course.
- For students sitting the Junior Certificate exam from **2015 onwards**, the Higher Level course is fully covered in *Active Maths 2*.

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